ON THE GOLDIE DIMENSION OF HEREDITARY RINGS AND MODULES

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ABSTRACT. We find a bound for the Goldie dimension of hereditary modules in terms of the cardinality of the generator sets of its quasi-injective hull. Several consequences are deduced. In particular, it is shown that every right hereditary module with countably generated quasi-injective hull is noetherian. Or that every right hereditary ring with finitely generated injective hull is artinian, thus answering a long standing open question posed by Dung, Gómez Pardo and Wisbauer.

1. INTRODUCTION.

Let R be an associative ring with identity. The (infinite) Goldie (or uniform) dimension of a right R-module M, Gdim(M), is defined to be the supremum of all cardinal numbers $\aleph$ such that there exists a direct sum $\oplus_I M_i \subseteq M$ of non-zero submodules of M with $|I| = \aleph$ (see e.g. [6]). This definition is based on lattice-theoretical properties of the set of submodules of M. Therefore, it can also be stated in terms of Lattice Theory, specifically in modular lattices (see [20]). This gives the context of an interesting common framework where it is possible to include, for example, the study of eigenvalues of Hermitian compact operators or the singular values of compact operators (see [14]). It is worth mentioning that these theories give examples of lattices of infinite Goldie dimension (concretely, countable) that are not covered by the classical notion of finite Goldie dimension.

A main question when dealing with infinite Goldie dimensions of modules (or modular lattices) is whether a module of Goldie dimension $\aleph$ must necessarily contain a direct sum of $\aleph$ non-zero submodules. In this case it is said that Gdim(M) is *attained in M*. The main result of [6] (see also [20]) asserts that Gdim(M) is attained whenever it is not a weakly inaccessible cardinal. Moreover, some examples are given

showing that for weakly inaccessible cardinals, Goldie dimensions may be not attained. These examples are based on constructions by Ërdos and Tarski for Boolean Algebras [9]. However, let us point out that the existence of a weakly inaccessible cardinal would imply the consistency of ZFC (Zermelo-Fraenkel Set Theory with the Axiom of Choice), even assuming the Generalized Continuum Hypothesis (see, e.g. [16],[19]). Therefore, due to Gödel Incompleteness Theorems, the existence of a weakly inaccessible cardinal cannot be proved in ZFC. Actually, the existence of this type of cardinals is an independent statement in ZFC.

Using certain combinatorial methods in Set Theory, commonly refereed to by many Ring theorists as "Tarski's Lemma" [23], it was proved in [13] that if M is a non-singular injective (or, more generally, a quasi-continuous) module which contains an essential finitely generated submodule and if $\mathrm{Gdim}(M) = \aleph$ is attained and infinite, then there exists a quotient of M that contains an infinite direct sum of $\aleph^+$ non-zero submodules, where $\aleph^+$ stands for the successor cardinal of $\aleph$. This technical result was then applied to assure the finiteness of the Goldie dimension of M under various assumptions. Consider, for example, when every quotient of M has countable Goldie dimension [13, Corollary 2.4].

In this paper, we continue this line of reseach by studying the Goldie dimension of non-singular modules over hereditary rings. We first observe that the Goldie dimension of a projective module P is closely related to the minimal cardinality of the generator sets of its submodules. This relation is essential for proving our main results. First we show that if P is a finitely generated hereditary module whose quasi-injective hull is $\aleph$-generated for some infinite cardinal number $\aleph$, then M cannot contain an independent family of $\aleph$ non-zero submodules. When $\aleph$ is not an inaccessible cardinal this means that $\mathrm{Gdim}(P) < \aleph$. Several consequences are derived from them. Among them, we show that every right hereditary module such that the quasi-injective hulls of its cyclic submodules are countably generated is a direct sum of noetherian modules. In particular, we get that every right hereditary ring with countably generated injective hull is right noetherian, thus extending results in [11] and [12]. Possibly the most interesting consequence is Corollary 13, where it is shown that every right hereditary ring with finitely generated injective envelope is right artinian. This corollary answers an open question posed in [8] and [11, Remark (b), p. 1033] that has remained open for fifteen years and whose motivation sterns back to Osofsky's work on hypercyclic rings as well as to an old characterization of (two-sided) hereditary Artinian QF-3 rings given in

[2, Theorem 3.2]. See [11, Corollary 6] and [12, Corollary 1.10] for some partial answers to this problem.

In Section 3 we extend our results to modules M over right hereditary rings such that the (quasi-)injective hull of M is $\aleph$-presented for some cardinal number $\aleph$ of cofinality ω (see Theorem 16). In particular we obtain that any countably presented (quasi-)injective module over such rings is a direct sum of uniform modules . Let us point out that, in order to get this theorem, we need to drop the nonsingularity hypothesis in M, a critical condition in this kind of results (see [17, 18] for a deep discussion about this question). On the other hand any cardinal number of cofinality ω is (both weakly and strongly) accessible and, therefore, Goldie dimension is always attained for these cardinals (see [6]).

We close the paper by discussing in Section 4 possible ways to extend our techniques to a more general question. Namely, whether any finitely generated module satisfying that every quotient is injective is a direct sum of uniforms (Conjecture 20). We also show that our results give partial positive answers to this other conjecture.

Throughout this paper all rings will be associative and with identity. Mod-R will stand for the category of right R-modules. Given a cardinal number $\aleph$, we will denote by $\aleph^+$, the successor cardinal of $\aleph$, i.e., the smallest cardinal number that is strictly greater than $\aleph$.

We refer to [1, 4, 16, 24] for any notion used in the text but not defined herein.

2. MAIN RESULTS.

Let $\aleph$ be cardinal number. A right R-module M is called $\aleph$-generated if there exists an epimorphism $p : R^{(\aleph)} \to M$. And M is said to be $\aleph$-presented if the kernel of this epimorphism is also $\aleph$-generated. Given a module M, we are going to denote by $Add[M]$ the full subcategory of Mod-R consisting on direct summands of direct sums of copies of M. The following easy lemma points out the close relation that there exists between the cardinality of the generator sets of projective modules and their decompositions as direct sums of non-zero submodules. Since any projective module is an element of $Add[R_R]$, we state the lemma in this more general form.

Lemma 1. *Let $\aleph$ be an infinite cardinal number, M, an $\aleph$-generated right R-module and $\aleph'$, a cardinal number with $\aleph' > \aleph$. Given an $N \in Add[M]$, the following conditions are equivalent:*

   (1) *N can be decomposed as a direct sum of $\aleph'$ non-zero direct summands*

(2) *Every generator set of N has cardinality at least $\aleph'$.*

In particular, given an uncountable cardinal number c, a projective module P is the direct sum of c non-zero submodules if and only if every generator set of P has cardinality at least c.

Proof. 1) $\Rightarrow$ 2) Assume that $N = \oplus_I N_i$ with $|I| = \aleph'$ and $N_i \neq 0$ for every $i \in I$. And let $\{x_k\}_K$ be a generator set of N. For every $k \in K$, there exists a finite subset $I_k \subseteq I$ such that $x_k \in \oplus_{i \in I_k} N_i$. Thus, $N = \sum_{k \in K} x_k R = \sum_{k \in K} (\oplus_{i \in I_k} N_i) = \oplus_{i \in (\cup_{k \in K} I_k)} N_i$. Therefore, $I = \cup_{k \in K} I_k$ and, as each I_k is finite, this means that $|K| \geq |I|$.

2) $\Rightarrow$ 1) Assume now that every generator set of N has cardinality at least $\aleph'$. By Kaplansky's Theorem [1, Theorem 26.1], N is a direct sum of $\aleph$-generated submodules, say $N = \oplus_I N_i$. Let us choose, for every $i \in I$, a generator set X_i of N_i of cardinality $\aleph$. Then $\cup_I X_i$ is a generator set of N. Our assumption implies that $|\cup_I X_i|$ must be at least $\aleph'$. And, as $\aleph < \aleph'$, this means that $|I| \geq \aleph'$ $\square$

Recall that a right module P is called hereditary when every submodule is projective (see e.g. [15]). In particular, every projective module over a hereditary ring is hereditary.

Corollary 2. *Let P be a hereditary module. If $\mathrm{Gdim}(P)$ is infinite, then it equals the minimum cardinal number $\aleph$ such that every submodule of P is $\aleph$-generated.*

Proof. As P is hereditary, every submodule of P is projective. Let us distinguish two possibilities. If $\aleph > \aleph_0$ then the result is a consequence of the above Lemma.

Assume now that $\aleph = \aleph_0$. Lemma 1 assures that $\mathrm{Gdim}(P) \leq \aleph$. Therefore, $\aleph_0 \leq \mathrm{Gdim}(P) \leq \aleph = \aleph_0$. $\square$

The following Lemma summarizes some well-known properties of hereditary modules. We are including a proof for the sake of completeness.

Lemma 3. *Let P be a hereditary right R-module. Then*

(1) *P is non-singular*
(2) *Given an infinite cardinal number $\aleph$, P is $\aleph$-generated (resp., finitely generated) iff it contains an essential $\aleph$-generated (resp. finitely generated) submodule.*
(3) *Any submodule of a direct sum of copies of P is a direct sum of submodules of P. In particular, any direct sum of copies of P is hereditary.*

Proof.   (1) Let $x \in P$, $x \neq 0$. Its right annihilator, $r_R(x)$, is the kernel of the homoeomorphism $f : R \to xR$ consisting on right multiplication by x. As $xR \subseteq P$, it is projective. Therefore, $r_R(x)$ is a direct summand of R_R and it cannot be an essential right ideal.

(2) Let N be an essential $\aleph$-generated submodule of P. As P is projective, there exists a splitting epimorphism $\pi : R^{(I)} \to P$ for some index set I. Let $u : P \to R^{(I)}$ such that $\pi \circ u = 1_P$. Since N is $\aleph$-generated (resp. finitely generated), there exists a subset $K \subseteq I$ of cardinality $\aleph$ (resp. finite) such that $u(N) \subseteq R^{(K)}$. Let $q : R^{(I)} \to R^{(K)}$ and $v : R^{(K)} \to R^{(I)}$ be the canonical projection and injection, respectively. Then $1_P - (\pi \circ v \circ q \circ u)$ is an endomorphism of P whose kernel is essential, since it contains N. Therefore, $1_P - (\pi \circ v \circ q \circ u) = 0$ (because P is nonsingular) and this means that P is a direct summand of $R^{(K)}$.

(3) This may be proven by following the arguments in [24, 39.7 (2)]. It is interesting, however, to point out that this is indeed the case even though the definition of hereditary modules used in [24] is not the one we use in this paper. We are following the original definition of hereditary modules which goes back to [15]. $\square$

Theorem 4. *Let $\{M_i\}_I$ be a family of modules. Then $\mathrm{Gdim}(\oplus_I M_i) = \sum_I \mathrm{Gdim}(M_i)$. In particular, for any module M and any infinite index set I, $\mathrm{Gdim}(M^{(I)}) = \max \{\mathrm{Gdim}(M), |I|\}$*

Proof. See [6, Theorem 13] $\square$

We recall that the cofinality of a cardinal number $\aleph$ is defined to be the least ordinal number α such that there exists an injective increasing map $f : \alpha \to \aleph$ that is cofinal in $\aleph$. I.e., such that for any ordinal number $\gamma < \aleph$ there exists an ordinal $\beta < \alpha$ with $f(\beta) \geq \gamma$ (see e.g. [19, Section 5.4]). The cofinality of $\aleph$ is always a cardinal number that we will denote by $\mathrm{cof}(\aleph)$. It is clear that $\mathrm{cof}(\aleph) \leq \aleph$. A cardinal number $\aleph$ is called regular if $\mathrm{cof}(\aleph) = \aleph$. Otherwise, $\aleph$ is called singular. An uncountable cardinal $\aleph$ is said to be (weakly) inaccessible if it is both regular and limit (i.e., it is not the successor of any other cardinal). The main result of [6] shows that $\mathrm{Gdim}(M)$ is always attained whenever it is not an inaccessible cardinal.

The following Theorem will be crucial for our first upper bound of the Goldie dimension of a hereditary module P in terms of its quasi-injective hull (Theorem 8 below).

Theorem 5. *Let P be a hereditary module, M a finitely generated submodule of P and $Q(M)$, the P-injective hull of M. If $Q(M)$ is $\aleph$-presented for some infinite cardinal number $\aleph$, then every submodule of M has a generator set with cardinality strictly smaller than $\aleph$.*

Proof. Assume on the contrary that there exists a submodule L of M such that every generator set of L has cardinality at least $\aleph$. We show next that this implies that L contains a direct sum of $\aleph$ nonzero submodules, say $\oplus_I L_i$. If $\aleph > \aleph_0$, it is clear by Lemma 1. And, if $\aleph = \aleph_0$, it is a consequence of the fact that hereditary modules of finite Goldie dimension are finitely generated by Lemma 3 (2). Moreover, let us realize that we can assume that each L_i is finitely generated and therefore, $\oplus_I L_i$ is $\aleph$-generated. And, adding a complement if necessary (see [1, 5.21]), we can also assume that $\oplus_I L_i$ is essential in M.

Let $Q(M)$ be the P-injective envelope of M. $Q(M)$ is $\aleph$-presented by assumption. So there exists an epimorphism $\pi : P^{(A)} \to Q(M)$ with $|A| \leq \aleph$ and $Ker(\pi)$, an $\aleph$-generated module. Using [13, Theorem 2.2], we deduce that there exists a submodule N of Q such that $\aleph^+$ is attained in Q/N. Let $X = \oplus_{j \in J} X_j$ be a direct sum of non-zero modules contained in Q/N with $|J| = \aleph^+$. And let $q : Q \to Q/N$ be the canonical projection. Then $(q \circ \pi)^{-1}(X)$ is a submodule of $P^{(A)}$ that cannot have a generator set of cardinality at most $\aleph$, since every generator set of X has clearly cardinality at least $\aleph^+$. Moreover, $(q \circ \pi)^{-1}(X)$ is projective, since $P^{(A)}$ is hereditary. Thus, $(q \circ \pi)^{-1}(X)$ is a direct sum of non-zero modules in cardinality $\aleph^+$, by Lemma 1. Say $(q \circ \pi)^{-1}(X) = \oplus_B Y_b$ with $|B| = \aleph^+$.

Let us choose an $\aleph$-generated submodule N of $P^{(A)}$ such that $\pi(N) = \oplus_I L_i$. Then $N + Ker(\pi)$ is essential in $(q \circ \pi)^{-1}(X)$ because $\oplus_I L_i$ is essential in M. But $N + Ker(\pi)$ is $\aleph$-generated, since so are N and $Ker(\pi)$. And this means that there exists a subset $B' \subseteq B$ of cardinality $\aleph < |B|$ such that $N + Ker(\pi) \subseteq \oplus_{B'} Y_b$. Let us pick an element $b \in B \setminus B'$. Then $Y_b \cap (N + Ker(\pi)) = 0$, which is a contradiction because $N + Ker(\pi)$ is essential in $(q \circ \pi)^{-1}(X)$. $\square$

Corollary 6. *Let $\aleph$ be a cardinal number and P, an $\aleph$-generated hereditary module. Let $\aleph'$ be an infinite cardinal number such that $\mathrm{cof}(\aleph') > \aleph$. If the quasi-injective hull of P is $\aleph'$-presented, then $\aleph'$ is not attained in P/L for any submodule L of P.*

Proof. Assume on the contrary that $\aleph'$ is attained in P/L for some submodule L of P. By Lemma 1 there exists a submodule of P/L (and thus, a submodule of P) satisfying that every generator set has cardinality at least $\aleph'$. Reasoning as in the above theorem, we deduce that $\aleph'$ is also attained in P.

Let $\{M_i\}_I$ be the set of finitely generated submodules of P. By Lemma 3, P is a direct sum of countably generated submodules of the M_i's, say $P = \oplus_A P_a$, with $|A| \leq \aleph < \text{cof}(\aleph')$. Thus, $\aleph'$ is attained in P_a, for some $a \in A$, by Lemma [6, Lemma 2]. But P_a is a submodule of M_i for some $i \in I$. This means that $\aleph'$ is attained in M_i and thus, M_i contains a submodule L verifying that any generator set has cardinality at least $\aleph'$, which is a contradiction with Theorem 5. $\qquad\square$

Corollary 7. *Let $\aleph$ a cardinal number and P an $\aleph$-generated hereditary module. Let $\aleph' > \aleph$ be an infinite cardinal number. If the quasi-injective hull Q of P is $\aleph'$-presented, then $\text{Gdim}(P/L) \leq \aleph'$ for every submodule L of P.*

Proof. Assume on the contrary that $\text{Gdim}(P/L) > \aleph'$ for some submodule L of P. Then $\text{Gdim}(P) \geq (\aleph')^+$ and thus, $\aleph^+$ is attained in P/L since it is not an inaccessible cardinal. Moreover, $\text{cof}((\aleph')^+) = \aleph^+ > (\aleph')$. But this contradicts Corollary 6. $\qquad\square$

Our next result improves Theorem 5 when the considered hereditary module is finitely generated.

Theorem 8. *Let P be a finitely generated hereditary module and $\aleph$, an infinite cardinal number. If the quasi-injective hull $Q(P)$ of P is $\aleph$-generated, then $\aleph$ is not attained in P/L for any submodule L of P.*

Proof. Assume on the contrary that $\aleph$ is attained in P/L for some submodule L of P. The $\aleph$ is attained in P by the same reason as in Theorem 5. Let $\oplus_I L_i$ be a direct sum of non-zero submodules of P with $|I| = \aleph$. And let $\pi : P^{(A)} \to Q(P)$ be an epimorphism with $|A| \leq \aleph$. Let K be the kernel of p. K is projective and, in particular, a direct sum of countably generated modules, say $K = \oplus_J K_j$. Let $\aleph' = \max\{\aleph, |J|\}$. We claim that P contains an infinite direct sum of $\aleph'$ non-zero submodules.

By Theorem 4, we know that

$$\text{Gdim}\left(P^{(A)}\right) = \sum_A \text{Gdim}(P) = |A| \cdot \text{Gdim}(P) = \max\{\aleph, \text{Gdim}(P)\}$$

and, as we are assuming that $\aleph \leq \text{Gdim}(P)$, we deduce that $\text{Gdim}(P) = \text{Gdim}\left(P^{(A)}\right) \geq \aleph'$. Let us check that $\aleph'$ is attained in P. If $\aleph' = \aleph$, this is obvious, since P contains $\oplus_I L_i$ with $|I| = \aleph$. So let us assume

that $\aleph' = |J| > \aleph$. If $|J|$ is not an inaccessible cardinal, then $|J|$ is attained in P by [6, Theorem 6]. Otherwise, $|J|$ equals its cofinality. And therefore, $cof\,(|J|) = |J| > \aleph$ is attained in P, since it is clearly attained in $P^{(A)}$ (see [6, Lemma 2]).

On the other hand, $Q(P)$ is $\aleph'$-presented. Therefore, $\aleph'$ cannot be attained in P by Theorem 5. That is a contradiction which shows that $\aleph$ cannot be attained in P/L. $\qquad\square$

Next corollary shows that Theorem 8 is particularly interesting when applied to finitely generated hereditary modules with countably generated quasi-injective hull.

Corollary 9. *Let P be a finitely generated hereditary R-module with countably generated quasi-injective hull. Then P is a noetherian module.*

Proof. Let N be any submodule of P. The above theorem shows that N must have finite Goldie dimension. Therefore, it is finitely generated by Lemma 3 (2). $\qquad\square$

In particular, we get the following corollary for right hereditary rings that extends results in [11, 12].

Corollary 10. *Let R be a right hereditary ring. If the injective envelope of R_R is countably generated, then R is right noetherian.*

We do not know whether Corollary 9 remains true for any hereditary right module P. However, our next proposition shows that this is the case when the quasi-injective hull of any cyclic submodule of P is countably generated.

Proposition 11. *Let P be a hereditary right R-module. If the quasi-injective hull of any cyclic submodule of P is countably generated, then P is a countable direct sum of noetherian modules.*

Proof. By Corollary 9, any cyclic submodule of P is noetherian. Let $\{N_i\}_{i \in I}$ be the family of all cyclic submodules of P. It is clear that $\oplus_I N_i$ is hereditary by Lemma 3(3). Moreover, P is a quotient of $\oplus_I N_i$ and thus, a direct summand. Finally, P is a direct sum of submodules of the M_i's, again by Lemma 3(3). $\qquad\square$

Remark 12. *Let us note that if $P = \oplus_I P_i$ is a direct sum of noetherian modules, then the Grothendieck category $\sigma[P]$ (see [24] for the definition) is locally noetherian (i.e., it has a generator set consisting on noetherian objects). As a module Q in $\sigma[P]$ is injective iff it is a P-injective module, we deduce that the quasi-injective hull of P is*

Σ-*quasi-injective. Therefore, the quasi-injective hull of the hereditary modules considered in the above corollary is* Σ-*quasi-injective.*

We close this Section by proving the following interesting corollary that gives a positive answer to the question posed by Dung in [4] and by Dung, Gómez Pardo and Wisbauer in [11].

Corollary 13. *Let R be a right hereditary ring. If $E(R_R)$ is finitely generated, then R_R is artinian.*

Proof. We know that R is right noetherian by the above corollary. The result now follows from [22, Theorem A]. □

3. Additional results.

The results given in the above section can only be applied to hereditary modules. In particular, projective modules over right hereditary rings. In this section we are going to show that, under certain additional hypothesis, the given arguments can be slightly modified in order to cover other situations. We begin by showing how to apply them to arbitrary non-singular modules over right hereditary rings.

Proposition 14. *Let R be a right hereditary ring, $\aleph$, an infinite cardinal number and M a non-singular finitely generated right R-module. If the (quasi-)injective hull $Q(M)$ of M is $\aleph$-presented, then $\aleph$ is not attained in M.*

Proof. Let us adapt the arguments given in the proof of Theorem 5. Assume otherwise that $\aleph$ is attained in M. Then M contains a direct sum of non-zero finitely generated modules, say $\oplus_I M_i$, with $|I| = \aleph$. We are assuming that $Q(M)$ is $\aleph$-presented. So there exists an epimorphism $\pi : R^{(B)} \to Q(M)$ with $|B| = \aleph$ such that $Ker(\pi)$ is $\aleph$-presented. Now all the arguments given in Theorem 5 apply to this setup. □

The assumption that M is nonsingular in the above proposition is essential in the proof. The reason is that otherwise we do not have uniqueness in (M-)injective hulls of submodules of M in $Q(M)$. And therefore, we cannot apply [13, Theorem 2.2] in our arguments (see [17, 18] for an interesting discussion on this problem). We are going to show that it is possible to drop this assumption when we choose a cardinal $\aleph$ with cofinality ω. Let us note that this situation has a particular interest since $\aleph_0$ has cofinality ω. First we will need a technical lemma.

Lemma 15. *Let $p : M \to N$ be a splitting epimorphism of right R-modules. If L is an essential submodule of M, then $p(L)$ is an essential*

submodule of N. In particular, if M contains an essential cyclic sub-module, then so does N.

Proof. As p is splitting, there exists a $u : N \to M$ such that $p \circ u = 1_M$. Let K be a non-zero submodule of N. Then $u(K)$ is a nonzero submodule of M. $L \cap u(K) \neq 0$ since L is essential in M. Therefore $0 \neq p(L \cap u(K)) \subseteq K \cap p(L)$. Thus, $p(L)$ is essential in N. $\square$

Theorem 16. *Let R be a right hereditary ring, $\aleph$, an infinite cardinal number of cofinality ω and M, a finitely generated right R-module. If the (quasi-)injective hull $Q(M)$ of M is $\aleph$-presented, then $\mathrm{Gdim}(M) < \aleph$.*

Proof. Assume on the contrary that $\mathrm{Gdim}(M) = \aleph$. As any cardinal number with cofinality ω is accessible, $\aleph$ is attained in M. Let $\oplus_A M_\alpha \subseteq M$ be a direct sum of nonzero submodules of M with $|A| = \aleph'$. And let $\pi : R^{(I)} \to Q(M)$ be an epimorphism with $|I| \leq \aleph$ and $Ker(\pi)$, $\aleph$-generated.

Let us fix, for any $\alpha \in A$, an (M-)injective hull Q_α of M_α within $Q(M)$. As $\mathrm{cof}(\aleph) = \omega$, Tarski's Lemma [23, Théorème 7] assures the existence of a subset $\mathcal{K} \subseteq \aleph^\omega$ with $|\mathcal{K}| > \aleph$ such that any $K \in \mathcal{K}$ has cardinality $\aleph_0$ and $K \cap K'$ is finite if $K \neq K'$. Let Q_K be an (M-)injective hull of $\oplus_K Q_\alpha$ in $Q(M)$ for any $K \in \mathcal{K}$. And let $Z = \sum_{\mathcal{K}} Q_K$.

We claim that Z is not $\aleph$-generated. Otherwise, there would exist a subset $\mathcal{A} \subseteq K$ of cardinality $\aleph$ such that $Z \subseteq \sum_{K \in \mathcal{A}} Q_K$. Therefore, there would exist an element $K_0 \in \mathcal{K} \setminus \mathcal{A}$ such that $Q_{K_0} \subseteq \sum_{K \in \mathcal{A}} Q_K$. As Q_{K_0} is a direct summand of $Q(M)$, there exists a splitting epimorphism $q : Q(M) \to Q_{K_0}$. And $M_{K_0} = q(M)$ is an essential finitely generated submodule of Q_{K_0} by Lemma 15. Therefore, there is a finite subset $\mathcal{A}' \subseteq \mathcal{A}$ such that $M_{K_0} \subseteq \sum_{K \in \mathcal{A}'} Q_K$.

As $|K_0| = \aleph_0$ and $K_0 \cap K$ is finite for any $K \neq K_0$, there exist a $k_0 \in K_0 \setminus \cup_{\mathcal{A}'} K$. And this means that $Q_{k_0} \cap \sum_{K \in \mathcal{A}'} Q_K \neq 0$, since as M_{k_0} is essential in Q_{k_0}. But this is a contradiction, because $\oplus_{K \in \mathcal{A}'} (\oplus_{k \in K} Q_k)$ is essential in $\sum_{K \in \mathcal{A}'} Q_K$ and $k_0 \notin K$ for any $K \in \mathcal{A}'$.

Therefore, we have shown that Z cannot be $\aleph$-generated. Now, we can use the same arguments as in Theorem 5 to get a contradiction. $\square$

In particular, we get the following corollary that extends results in [11].

Corollary 17. *Let R be a right hereditary ring and Q a countably presented quasi-injective right R-module. Then Q is a countable direct sum of uniform submodules.*

Proof. Using the arguments of [7, 10.1], we can write $Q = \oplus_I Q_i$, where I is countable and each E_i is the quasi-injective hull of a cyclic module. Now, the above theorem shows that each Q_i has finite Goldie dimension and therefore, it is a finite direct sum of uniform modules. $\qquad\square$

Corollary 18. *Let R be a right hereditary ring and E, a finitely presented injective module. Then every quotient of E is finite-dimensional.*

Proof. Every quotient of E is finitely generated and injective. Therefore, the result follows from the above corollary. $\qquad\square$

Remark 19. *We recall that a ring R is called right PCI if every cyclic right module is either free or injective (see [10]). It was proved in [10] that any right PCI ring is right hereditary. Moreover, Damiano [3] showed that any right PCI ring is right noetherian (see also [11]). The key fact in Damiano's proof is to show that any finitely presented injective module over a right hereditary ring has finite Goldie dimension. Therefore, the above corollary gives an alternative proof of Damiano's result.*

4. Final remarks.

Our arguments show that right hereditary rings with finitely generated injective hull have finite Goldie dimension and therefore, are right artinian. Thus solving the question posed in [8, 11]. However we do not know whether they can be extended to answer the following more general conjecture:

Conjecture 20. *(see e.g. [7]) Let E be a finitely generated injective module such that every quotient of E is injective. Then E is a direct sum of uniforms.*

Our results show that the answer is "yes" for countably presented injective modules over right hereditary rings. Furthermore, the next proposition shows that our arguments can be slightly modified in order to show that the conjecture is true for the class of rings having cardinality at most $2^{\aleph_0}$.

Proposition 21. *Let R be a ring of cardinality at most $2^{\aleph_0}$ and let E be a countably generated injective module. If every quotient of E is injective, then E is a direct sum of indecomposable modules.*

Proof. Assume on the contrary that E is not a direct sum of uniforms. Then $Gdim(E) \geq \aleph_0$. Let $\oplus_I E_i \subset E$ be a direct sum of nonzero injective submodules with $|I| = \aleph_0$. By [13, Theorem 2.2] there exists a submodule $L \subseteq E$ such that $\aleph_1$ is attained in E/L. Let $\oplus_J Q_j$ be a

direct sum on non-zero submodules of E/L with $|J| = \aleph_1$. As R is right hereditary, E/L is injective. Therefore, we can choose for any subset $X \subseteq J$, an injective envelope Q_X of $\oplus_{j \in X} Q_j$ within E/L. Clearly, $Q_X \neq Q_Y$ if $X \neq Y$.

On the other hand E/L is countably generated since it is a quotient of E. And thus, each E_X is also countably generated. Therefore, E/L is a countably generated module that contains at least $2^{\aleph_1}$ different countably generated submodules. But this is a contradiction, since $|R| \leq 2^{\aleph_0}$. $\square$

Let us finish the paper by pointing out that not even for finitely generated modules over right hereditary rings do we know the answer to Conjecture 20 when $|R| > 2^{\aleph_0}$. In fact, we do not even know a counterexample to the following more general question:

Question 22. *Let E be a finitely generated module such that any pure quotient is pure-injective. Is E a direct sum of indecomposable pure-injective modules?*

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